Conformal Deformation to Scalar Flat Metrics with Constant Mean Curvature on the Boundary in Higher Dimensions

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On a closed Riemannian manifold of dimension $n \geq 3$, every metric is conformal to a constant scalar curvature metric. This problem, called the Yamabe problem, was proved by Yamabe [20], Trudinger [19], Aubin [1] and Schoen [18].

To extend the conformal deformation problem to manifolds with boundary, Escobar proposed two types of formulations. Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ with boundary ∂M. We denote by R_g the scalar curvature of the manifold and by κ_g the mean curvature of the boundary. The first type is to find a metric $\tilde{g}$ in the conformal class of g such that $R_{\tilde{g}}$ is constant and $\kappa_{\tilde{g}}$ is zero. This was studied by Escobar [12] and recently by Brendle and the author [6].

The second type is to find a metric $\tilde{g}$ in the conformal class of g such that $R_{\tilde{g}}$ is zero and $\kappa_{\tilde{g}}$ is constant. This problem, as Escobar remarked [11], is a higher dimensional generalization of the Riemann mapping theorem. The problem is studied by Escobar [11], [13] and Marques [16], [17]. (For analysis background for both problems, see [9]).

In this paper, we will study the second formulation; that is the existence of a conformal metric with zero scalar curvature and constant mean curvature on the boundary. The problem turns out to be finding a critical point of the functional

$$E_g(\phi) = \frac{\int_M (\frac{4(n-1)}{n-2}|\nabla_g \phi|^2 + R_g \phi^2) dV_g + \int_{\partial M} 2\kappa_g \phi^2 d\sigma_g}{(\int_{\partial M} \phi^{\frac{2(n-1)}{n-2}} d\sigma_g)^{\frac{n-2}{n-1}}},$$

where ϕ is a positive smooth function on M. The exponent $\frac{2(n-1)}{n-2}$ is critical for the trace Sobolev embedding $H^1(M) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\partial M)$. This embedding is not compact and the functional E_g does not satisfy the Palais-Smale condition. For this reason, standard variational methods cannot be applied.

To study the problem, we consider the Sobolev quotient, introduced in [11],

$$\mathcal{Q}(M, \partial M, g) = \inf_{0<\phi \in C^\infty} E_g(\phi).$$

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This is known that $\mathcal{Q}(M, \partial M, g)$ is a conformal invariant and $\mathcal{Q}(M, \partial M, g) \leq \mathcal{Q}(B^n, \partial B^n)$, where $\mathcal{Q}(B^n, \partial B^n)$ is the Sobolev quotient of the unit ball B^n in $\mathbb{R}^n$ equipped with the flat metric. It was proved by Escobar that

Theorem 1. *(Escobar [11]) If $\mathcal{Q}(M, \partial M, g) < \mathcal{Q}(B^n, \partial B^n)$, then there exists a metric $\tilde{g}$ in the conformal class of g such that $R_{\tilde{g}}$ is zero and $\kappa_{\tilde{g}}$ is constant.*

For $n \geq 6$, when ∂M is not umbilic, Escobar showed that $\mathcal{Q}(M, \partial M, g) < \mathcal{Q}(B^n, \partial B^n)$. He also proved the inequality holds when $n = 3$, and when $n = 4, 5$ and ∂M is umbilic, provided M is not conformally equivalent to the unit ball. When $n = 4, 5$, and ∂M is not umbilic, Marques verified that the inequality holds.

Consequently, it remains to consider the case that $n \geq 6$ and ∂M is umbilic (some special case was considered in [16]). As in [4], [6], we denote by $\mathcal{Z}$ the set of points $p \in M$ such that

$$\limsup_{x \to p} d(p, x)^{2-d} |W_g|(x) = 0,$$

where $d = [\frac{n-2}{2}]$ and W_g is the Weyl tensor of g. We note that $p \in \mathcal{Z}$ if and only if $\nabla^m W_g(p) = 0$ for $m = 0, \cdots, d-2$. Moreover, the set $\mathcal{Z}$ is conformally invariant.

Our main result is

Theorem 2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$ with umbilic boundary. Suppose there exists a point $p \in \partial M$ such that $p \notin \mathcal{Z}$, then $\mathcal{Q}(M, \partial M, g) < \mathcal{Q}(B^n, \partial B^n)$. As a result, there exists a metric $\tilde{g}$ in the conformal class of g such that $R_{\tilde{g}}$ is zero and $\kappa_{\tilde{g}}$ is constant.*

We now discuss the case that $p \in \mathcal{Z}$ for all $p \in \partial M$. In Section 4, we consider a flux integral $\mathcal{I}(p, \delta)$ introduced in [6] in a small neighborhood of $p \in \partial M$. When $p \in \mathcal{Z}$, it was shown in [6] that $\lim_{\delta \to 0} \mathcal{I}(p, \delta)$ exists and is equal to a positive multiple of ADM mass of certain scalar flat asymptotically flat manifold; see Section 4. We reduce the case to positivity of mass.

Theorem 3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$ with umbilic boundary. Suppose there exists a point $p \in \partial M$ such that $p \in \mathcal{Z}$ and $\lim_{\delta \to 0} \mathcal{I}(p, \delta) > 0$, then $\mathcal{Q}(M, \partial M, g) < \mathcal{Q}(B^n, \partial B^n)$. As a result, there exists a metric $\tilde{g}$ in the conformal class of g such that $R_{\tilde{g}}$ is zero and $\kappa_{\tilde{g}}$ is constant.*

We give the outline of the proof. By Marques [16], we may choose conformal Fermi coordinates around a boundary point p. In these coordinates, we define

$$v_\epsilon = \left(\frac{\epsilon}{(\epsilon + x_n)^2 + \sum_{1 \leq a \leq n-1} x_a^2} \right)^{\frac{n-2}{2}}.$$

We note that v_ϵ is the extremal function for the sharp trace Sobolev inequality on the half plane; see [10], [2]. By conformal invariance, it holds

$$\mathcal{Q}(B^n, \partial B^n) \left(\int_{\partial \mathbb{R}^n_+} v_\epsilon^{\frac{2(n-1)}{n-2}} d\sigma \right)^{\frac{n-2}{n-1}} = \frac{4(n-1)}{n-2} \int_{\mathbb{R}^n_+} |\nabla v_\epsilon|^2 dx.$$

It is then understood that v_ϵ is the model function on $\mathbb{R}^n_+$.

We now consider the function $v_\epsilon + \psi$ defined in a small neighborhood of p, where ψ satisfies

$$\Delta\psi = \sum_{i,k=1}^{n} \left(\frac{n-2}{4(n-1)} v_\epsilon \partial_i \partial_k S_{ik} + \partial_k(\partial_i v_\epsilon S_{ik})\right) \quad \text{in } B_\delta \cap \mathbb{R}^n_+, \tag{1}$$

$$\partial_n \psi = -\frac{1}{2(n-1)} \partial_n v_\epsilon S_{nn} + \frac{n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon \psi \quad \text{on } B_\delta \cap \partial\mathbb{R}^n_+. \tag{2}$$

In the above equations, the tensor S_{ij} comes from applying the conformal killing operator to certain vector field we solve; see Section 2. The equation (1) corresponds to a linear approximation of the scalar curvature equation of $(v_\epsilon + \psi)^{\frac{4}{n-2}} g$. However, in our construction, the boundary condition (2) is not the linear approximation of the mean curvature equation of $(v_\epsilon + \psi)^{\frac{4}{n-2}} g$; the "linear mean curvature equation" should be

$$\partial_n \psi = \frac{n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon \psi.$$

We emphasize that the Sobolev quotient $\mathcal{Q}(M, \partial M, g)$ is normalized by the volume of the boundary (not the volume of the manifold). Our deformation of the metric does not fix the volume of the boundary locally. As a consequence, in order to get the energy functional small enough, the term $-\frac{1}{2(n-1)} \partial_n v_\epsilon S_{nn}$ is important because it cancels out *to the right order* the change of the volume of the boundary. This is the reason that the linear approximation of the mean curvature equation does not work here. This turns out to be the delicate part of the proof. Finally, to define a test function globally, we glue the function $v_\epsilon + \psi$ with the Green's function of the conformal Laplacian centered at p.

To show the above test function has the energy functional less than $\mathcal{Q}(B^n, \partial B^n)$, we use the method and techniques developed by Brendle [4] (see also [6]). In [4], these nice techniques were used to prove a convergence theorem for the Yamabe flow. In [6], these techniques were used to study the problem of first type described at the beginning. To be more precise, let $u_\epsilon = \epsilon^{\frac{n-2}{2}} (\epsilon^2 + |x|^2)^{-\frac{n-2}{2}}$. In [4], one considers the function $u_\epsilon + w$ in normal coordinates, where w satisfies $\Delta w + n(n+2) u_\epsilon^{\frac{4}{n-2}} w = \frac{n-2}{4(n-1)} u_\epsilon \partial_i \partial_k S_{ik} + \partial_k(\partial_i u_\epsilon S_{ik})$. In [6], one considers the function $u_\epsilon + w$ in Fermi coordinates together with the boundary condition $\partial_n w = 0$. We refer the readers to [3], [5], [15], [7], [8] for other related works concerning the Yamabe problem.

We introduce the notation in this paper. We denote by dx the volume element in $\mathbb{R}^n$, by $d\sigma$ the area element of a hypersurface in $\mathbb{R}^n$ and by $d\mu$ the area element of an $(n-2)$-dimensional surface in $\mathbb{R}^n$. We also denote by $\mathbb{R}^n_+$ the half plane $\{x : x_n \geq 0\}$. Let $B_r(x)$ be the ball of radius r centered at x. When x is at the origin, we simply denote by B_r.

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1 Background

Let $v_\epsilon(x) = \epsilon^{\frac{n-2}{2}}((\epsilon + x_n)^2 + \sum_{a=1}^{n-1} x_a^2)^{-\frac{n-2}{2}}$, $x \in \mathbb{R}^n_+$. The function v_ϵ satisfies

$$\Delta v_\epsilon = 0 \quad \text{for } x \in \mathbb{R}^n_+, \tag{3}$$

$$v_\epsilon \partial_i \partial_k v_\epsilon - \frac{n}{n-2} \partial_i v_\epsilon \partial_k v_\epsilon = -\frac{1}{n-2} |dv_\epsilon|^2 \delta_{ik} \quad \text{for } x \in \mathbb{R}^n_+, \tag{4}$$

and

$$\partial_n v_\epsilon = -(n-2) v_\epsilon^{\frac{n}{n-2}} \quad \text{for } x \ \in \partial\mathbb{R}^n_+. \tag{5}$$

By integration, we get

$$\int_{\mathbb{R}^n_+} |\nabla v_\epsilon|^2 dx = (n-2) \int_{\partial\mathbb{R}^n_+} v_\epsilon^{\frac{2(n-1)}{n-2}} d\sigma.$$

Moreover, v_ϵ satisfies the following inequalities:

$$\epsilon^{\frac{n-2}{2}} (\epsilon + |x|)^{-n+2} \leq v_\epsilon(x) \leq C(n) \epsilon^{\frac{n-2}{2}} (\epsilon + |x|)^{-n+2} \quad \text{for } x \in \mathbb{R}^n_+;$$

$$|\partial v_\epsilon|(x) \leq C(n) \epsilon^{\frac{n-2}{2}} (\epsilon + |x|)^{-n+1} \quad \text{for } x \in \mathbb{R}^n_+;$$

and

$$|v_\epsilon - \epsilon^{\frac{n-2}{2}} |x|^{-n+2}| \leq C(n) \epsilon^{\frac{n}{2}} |x|^{-n+1} \quad \text{for } x \in \mathbb{R}^n_+, \text{ and } |x| \geq 2\epsilon,$$

where $C(n)$ is a positive constant depending only on n.

Let V be a smooth vector field and H_{ik} be a trace-free symmetric two-tensor. We define

$S_{ik} = \partial_i V_k + \partial_k V_i - \frac{2}{n} div V \delta_{ik}$,
$T_{ik} = H_{ik} - S_{ik}$,
$P_{ik,l} = v_\epsilon \partial_l T_{ik} - \frac{2}{n-2} \partial_i v_\epsilon T_{kl} - \frac{2}{n-2} \partial_k v_\epsilon T_{il} + \frac{2}{n-2} \sum_{p=1}^n \partial_p v_\epsilon T_{ip} \delta_{kl} + \frac{2}{n-2} \sum_{p=1}^n \partial_p v_\epsilon T_{kp} \delta_{il}$,
$\psi = \partial_l v_\epsilon V_l + \frac{n-2}{2n} v_\epsilon div V$.

In [4], [6], a similar notation was introduced with v_ϵ replaced by u_ϵ.

The following formula is a revision of the formula in [4] Proposition 5, 6. The formula in [4] corresponds to the second variation of the scalar curvature on the sphere. Similarly, the formula here corresponds to the second variation of the scalar curvature on the ball in $\mathbb{R}^n$.

Proposition 1. *Let H_{ik} be a trace-free symmetric two-tensor, and V be a smooth vector field. Then ψ satisfies*

$$\Delta \psi = \sum_{i,k=1}^n \left(\frac{n-2}{4(n-1)} v_\epsilon \partial_i \partial_k S_{ik} + \partial_k (\partial_i v_\epsilon S_{ik}) \right). \tag{6}$$

Moreover,

$$\frac{1}{4}|P|^2 - \frac{1}{2}\sum_{i=1}^n |\sum_{k=1}^n (v_\epsilon \partial_k T_{ik} + \frac{2n}{n-2}\partial_k v_\epsilon T_{ik})|^2$$
$$= \sum_{i,k,l=1}^n (\frac{1}{4}v_\epsilon^2 \partial_l H_{ik}\partial_l H_{ik} - \frac{1}{2}v_\epsilon^2 \partial_k H_{ik}\partial_l H_{il} - 2v_\epsilon \partial_k v_\epsilon H_{ik}\partial_l H_{il} - \frac{2(n-1)}{n-2}\partial_k v_\epsilon \partial_l v_\epsilon H_{ik}H_{il})$$
$$+ \sum_{i,k=1}^n (-2v_\epsilon \psi \partial_i \partial_k H_{ik} + \frac{8(n-1)}{n-2}\partial_i v_\epsilon \partial_k \psi H_{ik}) - \frac{4(n-1)}{n-2}|d\psi|^2 + \sum_{i=1}^n \partial_i \xi_i,$$

where

$$\xi_i = \sum_{k=1}^n (2v_\epsilon \psi \partial_k H_{ik} - 2v_\epsilon \partial_k \psi H_{ik} - 2\partial_k v_\epsilon \psi H_{ik} - v_\epsilon \psi \partial_k S_{ik} + \partial_k (v_\epsilon \psi) S_{ik})$$
$$+ \sum_{k,l=1}^n (2v_\epsilon \partial_l v_\epsilon S_{kl} H_{ki} - \frac{1}{2}v_\epsilon^2 \partial_i S_{lk} H_{lk} + v_\epsilon^2 \partial_l S_{kl} H_{ki} + \frac{1}{4}v_\epsilon^2 \partial_i S_{lk} S_{lk} - \frac{1}{2}v_\epsilon^2 \partial_k S_{lk} S_{il})$$
$$+ \sum_{k,l=1}^n (-v_\epsilon \partial_k v_\epsilon S_{lk} S_{il} - \frac{2}{n-2}v_\epsilon \partial_k v_\epsilon T_{lk} T_{li}) + \frac{4(n-1)}{n-2}(-\sum_{k=1}^n \partial_k v_\epsilon \psi S_{ik} + \psi \partial_i \psi).$$

Proof. Since the proof is similar, we only point out the difference. In [4] Proposition 5, it was shown that

$$\begin{aligned} v_\epsilon \partial_i \partial_k S_{ik} + \frac{4(n-1)}{n-2}\partial_k(\partial_i v_\epsilon S_{ik}) &= \frac{4(n-1)}{n-2}\Delta(\sum_{l=1}^n \partial_l v_\epsilon V_l + \frac{n-2}{2n}v_\epsilon divV) \\ &- \frac{4(n-1)}{n-2}(\sum_{l=1}^n \partial_l \Delta v_\epsilon V_l + \frac{n+2}{2n}\Delta v_\epsilon divV) \end{aligned}$$

(with v_ϵ replaced by u_ϵ but the formula holds in general). By (3), then (6) follows.

For the second identity, by [4] Proposition 5, it holds

$$\begin{aligned} &\frac{1}{4}v_\epsilon^2|\partial T|^2 - \frac{1}{2}v_\epsilon^2|divT|^2 - \sum_{i,k,l=1}^n (2v_\epsilon \partial_k v_\epsilon T_{ik}\partial_l T_{il} + \frac{2(n-1)}{n-2}\partial_k v_\epsilon \partial_l v_\epsilon T_{ik}T_{il}) \\ = \; & I_1 - 2I_2 + I_3, \end{aligned}$$

where

$$I_1 = \sum_{i,k,l=1}^n (\frac{1}{4}v_\epsilon^2 \partial_l H_{ik}\partial_l H_{ik} - \frac{1}{2}v_\epsilon^2 \partial_k H_{ik}\partial_l H_{il} - 2v_\epsilon \partial_k v_\epsilon H_{ik}\partial_l H_{il} - \frac{2(n-1)}{n-2}\partial_k v_\epsilon \partial_l v_\epsilon H_{ik}H_{il}),$$

$$
\begin{aligned}
I_2 &= \sum_{i,k=1}^{n}(v_\epsilon\psi\partial_i\partial_k H_{ik} - \frac{4(n-1)}{n-2}\partial_i v_\epsilon\partial_k\psi H_{ik} - \partial_i(v_\epsilon\psi\partial_k H_{ik}) + \partial_k(v_\epsilon\partial_i\psi H_{ik}))\\
&+ \sum_{i,k=1}^{n}\partial_k(\partial_i v_\epsilon\psi H_{ik}) + \sum_{i,k,l=1}^{n}(\frac{1}{4}\partial_l(v_\epsilon^2\partial_l S_{ik}H_{ik}) - \frac{1}{2}\partial_k(v_\epsilon^2\partial_l S_{il}H_{ik}) - \partial_k(v_\epsilon\partial_l v_\epsilon S_{il}H_{ik}))\\
&+ \sum_{i,k,l=1}^{n}(v_\epsilon\partial_k\partial_l v_\epsilon - \frac{n}{n-2}\partial_k v_\epsilon\partial_l v_\epsilon)(\partial_l V_i - \partial_i V_l)H_{ik}\\
&- \sum_{i,k,l=1}^{n}\partial_l[(v_\epsilon\partial_i\partial_k v_\epsilon - \frac{n}{n-2}\partial_i v_\epsilon\partial_k v_\epsilon)V_l]H_{ik},
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \sum_{i,k=1}^{n}(v_\epsilon\psi\partial_i\partial_k S_{ik} - \frac{4(n-1)}{n-2}\partial_i v_\epsilon\partial_k\psi S_{ik} - \partial_i(v_\epsilon\psi\partial_k S_{ik}) + \partial_k(v_\epsilon\partial_i\psi S_{ik}))\\
&+ \sum_{i,k=1}^{n}\partial_k(\partial_i v_\epsilon\psi S_{ik}) + \sum_{i,k,l=1}^{n}(\frac{1}{4}\partial_l(v_\epsilon^2\partial_l S_{ik}S_{ik}) - \frac{1}{2}\partial_k(v_\epsilon^2\partial_l S_{il}S_{ik}) - \partial_k(v_\epsilon\partial_l v_\epsilon S_{il}S_{ik}))\\
&+ \sum_{i,k,l=1}^{n}(v_\epsilon\partial_k\partial_l v_\epsilon - \frac{n}{n-2}\partial_k v_\epsilon\partial_l v_\epsilon)(\partial_l V_i - \partial_i V_l)S_{ik}\\
&- \sum_{i,k,l=1}^{n}\partial_l[(v_\epsilon\partial_i\partial_k v_\epsilon - \frac{n}{n-2}\partial_i v_\epsilon\partial_k v_\epsilon)V_l]S_{ik}.
\end{aligned}
$$

And in [4] Proposition 6, it holds

$$
\begin{aligned}
&\frac{1}{4}v_\epsilon^2|\partial T|^2 - \frac{1}{2}v_\epsilon^2|divT|^2 - \sum_{i,k,l=1}^{n}(2v_\epsilon\partial_k v_\epsilon T_{ik}\partial_l T_{il} + \frac{2(n-1)}{n-2}\partial_k v_\epsilon\partial_l v_\epsilon T_{ik}T_{il})\\
&= \frac{1}{4}|P|^2 - \frac{1}{2}\sum_{i=1}^{n}|\sum_{k=1}^{n}(v_\epsilon\partial_k T_{ik} + \frac{2n}{n-2}\partial_k v_\epsilon T_{ik})|^2 - \frac{2}{(n-2)^2}|\partial v_\epsilon|^2|T|^2\\
&+ \sum_{i,k,l=1}^{n}(-\frac{2}{n-2}(v_\epsilon\partial_k\partial_l v_\epsilon - \frac{n}{n-2}\partial_k v_\epsilon\partial_l v_\epsilon)T_{ik}T_{il} + \frac{2}{n-2}\partial_l(v_\epsilon\partial_k v_\epsilon T_{ik}T_{il}))
\end{aligned}
\tag{7}
$$

(with v_ϵ replaced by u_ϵ but the formula holds in general). Using (4) in I_2, (4) and (6) in I_3 and using (4) in (7) give the identity.

□

2 Construction

We first state some properties about conformal Fermi coordinates that we will use later. Then we construct the correction term ψ and compute some formulas on the boundary. Let $n \geq 6$. We assume ∂M is totally geodesic.

In this section, we assume g is the metric in conformal Fermi coordinates. We write $g = \exp h$. By Marques [16], we have $tr\, h(x) = O(|x|^{2d+2})$ for $x \in \mathbb{R}^n_+$, where $d = [\frac{n-2}{2}]$. Moreover, $h_{in}(x) = 0$ for $x \in \mathbb{R}^n_+$ and $i = 1, \cdots, n$. We also have $\partial_n h_{ab}(x) = \sum_{i=1}^n h_{ai}(x)x_i = 0$ for $x \in \partial\mathbb{R}^n_+$ and $a, b = 1, \cdots, n-1$. In this case, $\det g(x) = 1 + O(|x|^{2d+2})$ for $x \in \mathbb{R}^n_+$.

Let H_{ij} be the Taylor expansion of h_{ij} up to the order d

$$H_{ij} = \sum_{2 \le |\alpha| \le d} h_{ij,\alpha}(0)x^\alpha,$$

where α is a multi-index. Then $h_{ik} = H_{ik} + O(|x|^{d+1})$. It follows that

$$tr H(x) = H_{in}(x) = 0 \quad \text{for all } x \in \mathbb{R}^n_+ \text{ and } i = 1, \cdots, n,$$

and

$$\partial_n H_{ab}(x) = \sum_{i=1}^n H_{ai}(x)x_i = 0 \quad \text{for all } x \in \partial\mathbb{R}^n_+ \text{ and } a, b = 1, \cdots, n-1.$$

We define algebraic Schouten tensor and algebraic Weyl tensor of H_{ij} as in [4]:

$$A_{ij} = \partial_i\partial_m H_{mj} + \partial_m\partial_j H_{im} - \Delta H_{ij} - \tfrac{1}{n-1}\partial_m\partial_p H_{mp}\delta_{ij},$$
$$Z_{ijkl} = \partial_i\partial_k H_{jl} - \partial_i\partial_l H_{jk} - \partial_j\partial_k H_{il} + \partial_j\partial_l H_{ik} + \tfrac{1}{n-2}(A_{jl}\delta_{ik} - A_{jk}\delta_{il} - A_{il}\delta_{jk} + A_{ik}\delta_{jl}).$$

Proposition 2. *[6] If $Z_{ijkl} = 0$ for all $x \in \mathbb{R}^n_+$, then $H_{ij} = 0$ for all $x \in \mathbb{R}^n_+$.*

Proposition 3. *[6] The scalar curvature R_g satisfies*

$$|R_g - \partial_i\partial_k H_{ik}| \le C\sum_{i,j}\sum_{2 \le |\alpha| \le d} |h_{ij,\alpha}||x|^{|\alpha|} + C|x|^{d-1},$$

and

$$\begin{aligned} & |R_g - \partial_i\partial_k h_{ik} + \partial_k(H_{ik}\partial_l H_{il}) - \frac{1}{2}\partial_k H_{ik}\partial_l H_{il} + \frac{1}{4}\partial_l H_{ik}\partial_l H_{ik}| \\ \le\ & C\sum_{i,j}\sum_{2 \le |\alpha| \le d} |h_{ij,\alpha}|^2|x|^{2|\alpha|} + C\sum_{i,j}\sum_{2 \le |\alpha| \le d} |h_{ij,\alpha}||x|^{|\alpha|+d-1} + C|x|^{2d} \end{aligned}$$

for $|x|$ sufficiently small.

Let V be a smooth vector field. We next define as in Section 1 that

$$S_{ik} = \partial_i V_k + \partial_k V_i - \tfrac{2}{n} div V\delta_{ik},$$
$$T_{ik} = H_{ik} - S_{ik},$$
$$P_{ik,l} = v_\epsilon\partial_l T_{ik} - \tfrac{2}{n-2}\partial_i v_\epsilon T_{kl} - \tfrac{2}{n-2}\partial_k v_\epsilon T_{il} + \tfrac{2}{n-2}\textstyle\sum_{p=1}^n \partial_p v_\epsilon T_{ip}\delta_{kl} + \tfrac{2}{n-2}\sum_{p=1}^n \partial_p v_\epsilon T_{kp}\delta_{il}.$$

Proposition 4. *Let V be a smooth vector field. Then*

$$\sum_{i,j}\sum_{2\leq|\alpha|\leq d}|h_{ij,\alpha}|^2\epsilon^{n-2}\int_{B_\delta\cap\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|+2-2n}dx\leq C(n)\int_{B_\delta\cap\mathbb{R}^n_+}|P|^2dx$$

for $\delta\geq 2\epsilon>0$.

Proof. In [4] Proposition 9, it was shown that

$$\begin{aligned}&\sum_{i,j,k,l=1}^{n}\{\partial_j(\partial_l T_{ik}-\frac{2}{n-2}v_\epsilon^{-1}\partial_k v_\epsilon T_{il})+\frac{2}{n-2}v_\epsilon^{-1}\partial_k v_\epsilon(\partial_j T_{il}-\frac{2}{n-2}v_\epsilon^{-1}\partial_i v_\epsilon T_{jl})\\&+\frac{2}{n-2}v_\epsilon^{-2}(v_\epsilon\partial_j\partial_k v_\epsilon-\frac{n}{n-2}\partial_j v_\epsilon\partial_k v_\epsilon)T_{il}+\frac{4}{(n-2)^2}v_\epsilon^{-2}\partial_k v_\epsilon(\partial_i v_\epsilon T_{jl}+\partial_j v_\epsilon T_{il})\}Z_{ijkl}\\&=\sum_{i,j,k,l=1}^{n}\partial_j\partial_l H_{ik}Z_{ijkl}\end{aligned}$$

(with v_ϵ replaced by u_ϵ but the formula holds in general). Then by (4), we have

$$\sum_{i,j,k,l=1}^{n}(\partial_j(v_\epsilon^{-1}P_{ik,l})Z_{ijkl}+\frac{2}{n-2}v_\epsilon^{-2}\partial_k v_\epsilon P_{il,j}Z_{ijkl})=\frac{1}{4}|Z|^2.$$

From this, the assertion follows easily by the proof in [6] Proposition 7 and Corollary 8 using $\epsilon^{\frac{n-2}{2}}(\epsilon+|x|)^{-n+2}\leq v_\epsilon(x)\leq C(n)\epsilon^{\frac{n-2}{2}}(\epsilon+|x|)^{-n+2}$ and $|\partial v_\epsilon|(x)\leq C(n)\epsilon^{\frac{n-2}{2}}(\epsilon+|x|)^{-n+1}$. □

We next construct the correction term ψ. We fix a positive smooth function $\eta(t)$ such that $\eta(t)=1$ for $t\leq\frac{4}{3}$ and $\eta(t)=0$ for $t\geq\frac{5}{3}$. For $\delta>0$, we define $\eta_\delta(x)=\eta(\frac{|x|}{\delta})$, $x\in\mathbb{R}^n_+$. Notice that $\partial_n\eta_\delta(x)=0$ for all $x\in\partial\mathbb{R}^n_+$. By Proposition 12 in Appendix, there exists a smooth vector field V which solves

$$\begin{cases}\sum_{k=1}^n\partial_k[v_\epsilon^{\frac{2n}{n-2}}(\eta_\delta H_{ik}-\partial_i V_k-\partial_k V_i+\frac{2}{n}divV\delta_{ik})]=0 & in\,\mathbb{R}^n_+\\ \partial_n V_a=0 & on\,\partial\mathbb{R}^n_+\\ V_n=0 & on\,\partial\mathbb{R}^n_+\end{cases}\tag{8}$$

for $i=1,\cdots,n$ and $a=1,\cdots,n-1$. Moreover, V satisfies

$$|\partial^\beta V^{(\epsilon,\delta)}(x)|\leq C(n,|\beta|)\sum_{i,k=1}^{n}\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|(\epsilon+|x|)^{|\alpha|+1-|\beta|}.\tag{9}$$

By the equation,

$$\sum_{k=1}^{n}(v_\epsilon\partial_k T_{ik}+\frac{2n}{n-2}\partial_k v_\epsilon T_{ik})=0\tag{10}$$

for $x \in B_\delta \cap \mathbb{R}^n_+$ and $i = 1, \cdots, n$. We next define

$$\psi = \sum_{l=1}^{n} \partial_l v_\epsilon V_l + \frac{n-2}{2n} v_\epsilon divV.$$

Proposition 5. *It holds* $S_{an}(x) = 0$,

$$\partial_n S_{nn}(x) = -\frac{2n}{n-2} v_\epsilon(x)^{-1} \partial_n v_\epsilon(x) S_{nn}(x) = 2n\, v_\epsilon(x)^{\frac{2}{n-2}} S_{nn}(x),$$

and

$$\partial_n S_{ab}(x) = -\frac{2n}{n-1} v_\epsilon(x)^{\frac{2}{n-2}} S_{nn}(x) \delta_{ab}$$

for $x \in \partial\mathbb{R}^n_+$ *and* $a, b = 1, \cdots, n-1$. *As a consequence, for* $x \in \partial\mathbb{R}^n_+$,

$$\partial_n \psi(x) = -\frac{1}{2(n-1)} \partial_n v_\epsilon(x) S_{nn}(x) + \frac{n}{n-2} v_\epsilon(x)^{-1} \partial_n v_\epsilon(x) \psi(x).$$

Proof. By assumptions, $V_n = \partial_n V_a = 0$ for $x \in \partial\mathbb{R}^n_+$ and $a = 1, \cdots, n-1$. Thus, $S_{na} = T_{na} = \partial_n V_a - \partial_a V_n = 0$ on $\partial\mathbb{R}^n_+$ for $a = 1, \cdots, n-1$ and

$$\partial_n \partial_a V_b = 0 \quad \text{for } x \in \partial\mathbb{R}^n_+ \text{ and } a, b = 1, \cdots, n-1. \tag{11}$$

We next consider the equation (8). It gives $\sum_{k=1}^n (v_\epsilon \partial_k(\eta_\delta H_{nk} - S_{nk}) + \frac{2n}{n-2} \partial_k v_\epsilon (\eta_\delta H_{nk} - S_{nk})) = 0$. Since $H_{nk}(x) = 0$ for all $x \in \mathbb{R}^n_+$ and $k = 1, \cdots, n$, we have

$$\sum_{k=1}^{n} (v_\epsilon \partial_k S_{nk} + \frac{2n}{n-2} \partial_k v_\epsilon S_{nk}) = 0.$$

for all $x \in \mathbb{R}^n_+$. Therefore, using (5)

$$\partial_n S_{nn} = -\sum_{a=1}^{n-1} \partial_a S_{na} - \frac{2n}{n-2} v_\epsilon^{-1} \sum_{k=1}^{n} \partial_k v_\epsilon S_{nk} = -\frac{2n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon S_{nn} = 2n v_\epsilon^{\frac{2}{n-2}} S_{nn}.$$

Moreover, by (11), it follows that

$$\begin{aligned}
\partial_n S_{ab} &= \partial_n \partial_a V_b + \partial_n \partial_b V_a - \frac{2}{n} \partial_n divV \delta_{ab} = -\frac{2}{n} \partial_n \partial_n V_n \delta_{ab} \\
&= -\frac{1}{n-1} (2\partial_n V_n - \frac{2}{n} \partial_n divV) \delta_{ab} = -\frac{1}{n-1} \partial_n S_{nn} \delta_{ab} = -\frac{2n}{n-1} v_\epsilon^{\frac{2}{n-2}} S_{nn} \delta_{ab}.
\end{aligned}$$

We now compute $\partial_n \psi$.

$$\begin{aligned}
\partial_n \psi =& \sum_{i=1}^{n} (\partial_n \partial_i v_\epsilon V_i + \partial_i v_\epsilon \partial_n V_i) + \frac{n-2}{2n} \partial_n v_\epsilon divV + \frac{n-2}{2n} v_\epsilon \partial_n divV \\
=& \sum_{i=1}^{n} (\partial_n \partial_i v_\epsilon - \frac{n}{n-2} v_\epsilon^{-1} \partial_i v_\epsilon \partial_n v_\epsilon) V_i + \sum_{i=1}^{n} \partial_i v_\epsilon \partial_n V_i \\
&+ \frac{n}{n-2} (\sum_{i=1}^{n} \partial_i v_\epsilon V_i + \frac{n-2}{2n} v_\epsilon divV) v_\epsilon^{-1} \partial_n v_\epsilon - \frac{1}{n} divV \partial_n v_\epsilon + \frac{n-2}{2n} v_\epsilon \partial_n divV.
\end{aligned}$$

By (4) and $\partial_n divV = \frac{n}{2(n-1)}\partial_n S_{nn}$, we get

$$\begin{aligned}\partial_n\psi =&\frac{n}{n-2}(\sum_{i=1}^n \partial_i v_\epsilon V_i + \frac{n-2}{2n}v_\epsilon divV)v_\epsilon^{-1}\partial_n v_\epsilon - \frac{1}{n}divV\partial_n v_\epsilon\\ &-\frac{1}{n-2}|dv_\epsilon|^2 V_n + \sum_{i=1}^n \partial_i v_\epsilon \partial_n V_i + \frac{n-2}{4(n-1)}v_\epsilon \partial_n S_{nn}.\end{aligned}$$

Since $\partial_n S_{nn} = -\frac{2n}{n-2}v_\epsilon^{-1}\partial_n v_\epsilon S_{nn}$ and $\partial_n V_a = V_n = 0$ on $\partial\mathbb{R}^n_+$ for $a = 1, \cdots, n-1$, then

$$\begin{aligned}\partial_n\psi =&\frac{n}{n-2}\psi v_\epsilon^{-1}\partial_n v_\epsilon - \frac{1}{n}divV\partial_n v_\epsilon + \partial_n v_\epsilon \partial_n V_n - \frac{n}{2(n-1)}\partial_n v_\epsilon S_{nn}\\ =&\frac{n}{n-2}\partial_n v_\epsilon v_\epsilon^{-1}\psi - \frac{1}{2(n-1)}\partial_n v_\epsilon S_{nn}.\end{aligned}$$

□

Proposition 6. *Let ξ_i be defined as in Proposition 1. It follows for $x \in \partial\mathbb{R}^n_+$,*

$$\xi_n(x) = -\frac{n+2}{2(n-2)}v_\epsilon(x)\partial_n v_\epsilon(x)S_{nn}(x)^2 + \frac{4n(n-1)}{(n-2)^2}v_\epsilon(x)^{-1}\partial_n v_\epsilon(x)\psi(x)^2.$$

Proof. Since $H_{in} = 0$ for $i = 1, \cdots, n$ and $x \in \mathbb{R}^n_+$, and $S_{na} = T_{na} = 0$ for $a = 1, \cdots, n-1$ and $x \in \partial\mathbb{R}^n_+$, we have

$$\begin{aligned}\xi_n &= -\frac{1}{2}v_\epsilon^2\sum_{a,b=1}^{n-1}\partial_n S_{ab}H_{ab} - v_\epsilon\psi\partial_n S_{nn} + v_\epsilon\partial_n\psi S_{nn} + \partial_n v_\epsilon\psi S_{nn}\\ &+ \frac{1}{4}v_\epsilon^2(\sum_{a,b=1}^{n-1}\partial_n S_{ab}S_{ab} + \partial_n S_{nn}S_{nn}) - \frac{1}{2}v_\epsilon^2\partial_n S_{nn}S_{nn} - v_\epsilon\partial_n v_\epsilon S_{nn}S_{nn}\\ &- \frac{2}{n-2}v_\epsilon\partial_n v_\epsilon S_{nn}S_{nn} + \frac{4(n-1)}{n-2}(-\partial_n v_\epsilon\psi S_{nn} + \psi\partial_n\psi).\end{aligned}$$

By $\partial_n S_{nn} = -\frac{2n}{n-2}v_\epsilon^{-1}\partial_n v_\epsilon S_{nn}$ and $\partial_n S_{ab} = -\frac{2n}{n-1}v_\epsilon^{\frac{2}{n-2}}S_{nn}\delta_{ab}$, we get

$$\begin{aligned}\xi_n &= \frac{n}{n-1}v_\epsilon^2 v_\epsilon^{\frac{2}{n-2}}S_{nn}\sum_{a=1}^{n-1}H_{aa} + \frac{2n}{n-2}\psi\partial_n v_\epsilon S_{nn} + v_\epsilon\partial_n\psi S_{nn} + \partial_n v_\epsilon\psi S_{nn}\\ &- \frac{1}{2}v_\epsilon^2(\frac{n}{n-1}v_\epsilon^{\frac{2}{n-2}}S_{nn}\sum_{a=1}^{n-1}S_{aa} + \frac{n}{n-2}v_\epsilon^{-1}\partial_n v_\epsilon S_{nn}^2) + \frac{n}{n-2}v_\epsilon\partial_n v_\epsilon S_{nn}^2\\ &- v_\epsilon\partial_n v_\epsilon S_{nn}S_{nn} + \frac{4(n-1)}{n-2}(-\partial_n v_\epsilon\psi S_{nn} + \psi\partial_n\psi) - \frac{2}{n-2}v_\epsilon\partial_n v_\epsilon S_{nn}S_{nn}.\end{aligned}$$

Thus,

$$\begin{aligned}
\xi_n &= \frac{n}{n-1} v_\epsilon^2 v_\epsilon^{\frac{2}{n-2}} S_{nn} \sum_{a=1}^{n-1} H_{aa} - \psi \partial_n v_\epsilon S_{nn} + v_\epsilon \partial_n \psi S_{nn} \\
&- \frac{1}{2} v_\epsilon^2 \Big(\frac{n}{n-1} v_\epsilon^{\frac{2}{n-2}} S_{nn} \sum_{a=1}^{n-1} S_{aa} + \frac{n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon S_{nn}^2 \Big) + \frac{4(n-1)}{n-2} \psi \partial_n \psi.
\end{aligned}$$

By $\sum_{a=1}^{n-1} H_{aa} = \sum_{i=1}^{n} S_{ii} = 0$ and (5), we get

$$\begin{aligned}
\xi_n &= -\partial_n v_\epsilon \psi S_{nn} + v_\epsilon \partial_n \psi S_{nn} - \frac{1}{2} v_\epsilon \Big(\frac{n}{(n-1)(n-2)} \partial_n v_\epsilon S_{nn}^2 + \frac{n}{n-2} \partial_n v_\epsilon S_{nn}^2 \Big) \\
&+ \frac{4(n-1)}{n-2} \psi \partial_n \psi \\
&= -\partial_n v_\epsilon \psi S_{nn} + v_\epsilon \partial_n \psi S_{nn} - \frac{n^2}{2(n-1)(n-2)} v_\epsilon \partial_n v_\epsilon S_{nn}^2 + \frac{4(n-1)}{n-2} \psi \partial_n \psi.
\end{aligned}$$

Finally, by $\partial_n \psi = -\frac{1}{2(n-1)} \partial_n v_\epsilon S_{nn} + \frac{n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon \psi$, we arrive at

$$\begin{aligned}
\xi_n &= -\partial_n v_\epsilon \psi S_{nn} + v_\epsilon \Big(-\frac{1}{2(n-1)} \partial_n v_\epsilon S_{nn} + \frac{n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon \psi \Big) S_{nn} \\
&- \frac{n^2}{2(n-1)(n-2)} v_\epsilon \partial_n v_\epsilon S_{nn}^2 + \frac{4(n-1)}{n-2} \psi \Big(-\frac{1}{2(n-1)} \partial_n v_\epsilon S_{nn} + \frac{n}{n-2} v_\epsilon^{-1} \partial_n v_\epsilon \psi \Big) \\
&= -\frac{n+2}{2(n-2)} v_\epsilon \partial_n v_\epsilon S_{nn}^2 + \frac{4n(n-1)}{(n-2)^2} v_\epsilon^{-1} \partial_n v_\epsilon \psi^2.
\end{aligned}$$

$\square$

3 Main estimates

In this section, we assume g is the metric in conformal Fermi coordinates as described in Section 2. Suppose V is a smooth vector field which satisfies (8) and (9). We adopt the notation in Section 2.

Proposition 7. *There exist positive numbers θ, C and δ_0 such that*

$$\begin{aligned}
&\int_{B_\delta\cap\mathbb{R}^n_+}(\frac{4(n-1)}{n-2}|d(v_\epsilon+\psi)|_g^2+R_g(v_\epsilon+\psi)^2)dx\\
&\leq 4(n-1)\int_{B_\delta\cap\partial\mathbb{R}^n_+} v_\epsilon^{\frac{2}{n-2}}(v_\epsilon^2+2v_\epsilon\psi+\frac{n}{n-2}\psi^2-\frac{n-2}{8(n-1)^2}v_\epsilon^2|S_{nn}|^2)d\sigma\\
&+\int_{\partial B_\delta\cap\mathbb{R}^n_+}\sum_{i=1}^n(\frac{4(n-1)}{n-2}v_\epsilon\partial_i v_\epsilon+v_\epsilon^2\partial_k h_{ik}-\partial_k v_\epsilon^2 h_{ik})\frac{x_i}{|x|}d\sigma\\
&-\theta\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|^2\epsilon^{n-2}\int_{B_\delta\cap\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|+2-2n}dx\\
&+C\epsilon^{n-2}\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|\delta^{|\alpha|+2-n}+C\epsilon^{n-2}\delta^{2d+4-n}
\end{aligned}$$

for $0<2\epsilon\leq\delta\leq\delta_0$, where $\theta=\theta(n)$, $C=C(n,g)$ and $\delta_0=\delta_0(n,g)$.

Proof. We write

$$\frac{4(n-1)}{n-2}|d(v_\epsilon+\psi)|_g^2+R_g(v_\epsilon+\psi)^2=\frac{4(n-1)}{n-2}|\partial v_\epsilon|^2+J_1+J_2+J_3+J_4,$$

where

$$\begin{aligned}
J_1=&\frac{8(n-1)}{n-2}\sum_{i=1}^n\partial_i v_\epsilon\partial_i\psi+\sum_{i,k=1}^n(-\frac{4(n-1)}{n-2}\partial_i v_\epsilon\partial_k h_{ik}+v_\epsilon^2\partial_i\partial_k h_{ik})\\
&-\sum_{i,k,l=1}^n(v_\epsilon^2\partial_k(H_{ik}\partial_l H_{il})+\partial_k v_\epsilon^2 H_{ik}\partial_l H_{il}),
\end{aligned}$$

$$\begin{aligned}
J_2=&\sum_{i,k,l=1}^n(-\frac{1}{4}v_\epsilon^2\partial_l H_{ik}\partial_l H_{ik}+\frac{1}{2}v_\epsilon^2\partial_k H_{ik}\partial_l H_{il}+\partial_k v_\epsilon^2 H_{ik}\partial_l H_{il}+\frac{2(n-1)}{n-2}\partial_k v_\epsilon\partial_l v_\epsilon H_{ik}H_{il})\\
&+\sum_{i,k=1}^n(2v_\epsilon\psi\partial_i\partial_k H_{ik}-\frac{8(n-1)}{n-2}\partial_i v_\epsilon\partial_k\psi H_{ik})+\frac{4(n-1)}{n-2}|d\psi|^2,
\end{aligned}$$

$$\begin{aligned}
J_3=&\frac{4(n-1)}{n-2}\sum_{i,k=1}^n(g^{ik}-\delta_{ik}+h_{ik}-\frac{1}{2}\sum_{l=1}^n H_{il}H_{kl})\partial_i v_\epsilon\partial_k v_\epsilon\\
&+(R_g-\sum_{i,k=1}^n\partial_i\partial_k h_{ik}+\sum_{i,k,l=1}^n\partial_k(H_{ik}\partial_l H_{il})-\frac{1}{2}(divH)^2+\frac{1}{4}|\partial H|^2)v_\epsilon^2,
\end{aligned}$$

and

$$J_4 = \frac{8(n-1)}{n-2} \sum_{i,k=1}^n (g^{ik} - \delta_{ik} + H_{ik}) \partial_i v_\epsilon \partial_k \psi + 2(R_g - \sum_{i,k=1}^n \partial_i \partial_k H_{ik}) v_\epsilon \psi$$
$$+ R_g \psi^2 + \frac{4(n-1)}{n-2} \sum_{i,k=1}^n (g^{ik} - \delta_{ik}) \partial_i \psi \partial_k \psi.$$

We compute

$$J_1 = \frac{8(n-1)}{n-2} \sum_{i=1}^n \partial_i (\partial_i v_\epsilon \psi) - \frac{8(n-1)}{n-2} \Delta v_\epsilon \psi + \sum_{i,k=1}^n (\partial_i (v_\epsilon^2 \partial_k h_{ik}) - \partial_k (\partial_i v_\epsilon^2 h_{ik}))$$
$$+ 2 \sum_{i,k=1}^n (v_\epsilon \partial_i \partial_k v_\epsilon - \frac{n}{n-2} \partial_i v_\epsilon \partial_k v_\epsilon) h_{ik} - \sum_{i,k,l=1}^n \partial_k (v_\epsilon^2 H_{ik} \partial_l H_{il}).$$

By (3) and (4),

$$J_1 \leq \frac{8(n-1)}{n-2} \sum_{i=1}^n \partial_i (\partial_i v_\epsilon \psi) + \sum_{i,k=1}^n (\partial_i (v_\epsilon^2 \partial_k h_{ik}) - \partial_k (\partial_i v_\epsilon^2 h_{ik})) - \sum_{i,k,l=1}^n \partial_k (v_\epsilon^2 H_{ik} \partial_l H_{il})$$
$$+ C \epsilon^{n-2} (\epsilon + |x|)^{2d+4-2n}.$$

Thus, integrating J_1 over $B_\delta \cap \mathbb{R}^n_+$ and using (9),

$$\begin{aligned} \int_{B_\delta \cap \mathbb{R}^n_+} J_1 dx \ \ &\leq \ \ \int_{\partial B_\delta \cap \mathbb{R}^n_+} \sum_{i,k=1}^n (v_\epsilon^2 \partial_k h_{ik} - \partial_k v_\epsilon^2 h_{ik}) \frac{x_i}{|x|} d\sigma - \int_{B_\delta \cap \partial \mathbb{R}^n_+} \frac{8(n-1)}{n-2} \partial_n v_\epsilon \psi d\sigma \\ &+ \ \ C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}| \delta^{|\alpha|+2-n} \epsilon^{n-2} + C \delta^{2d+4-n} \epsilon^{n-2}. \end{aligned}$$

For J_2, we first note that by Proposition 1 and (10), $J_2 = -\frac{1}{4}|P|^2 + \sum_{i=1}^n \partial_i \xi_i$. And by (9)

$$\int_{B_\delta \cap \mathbb{R}^n_+} \xi_i \frac{x_i}{|x|} d\sigma \leq C \sum_{i,k=1}^n \sum_{2 \leq |\alpha| \leq d} |h_{ik,\alpha}|^2 \delta^{2|\alpha|+2-n} \epsilon^{n-2}.$$

Moreover, by Proposition 4 there exists $\theta > 0$ such that

$$8\theta \sum_{i,j} \sum_{2 \leq |\alpha| \leq d} |h_{ij,\alpha}|^2 \epsilon^{n-2} \int_{B_\delta \cap \mathbb{R}^n_+} (\epsilon + |x|)^{2|\alpha|+2-2n} dx \leq \int_{B_\delta \cap \mathbb{R}^n_+} |P|^2 dx.$$

Hence, using Proposition 6

$$\begin{aligned}
\int_{B_\delta\cap\mathbb{R}^n_+} J_2 dx &= -\int_{B_\delta\cap\mathbb{R}^n_+}\frac{1}{4}|P|^2dx+\int_{\partial B_\delta\cap\mathbb{R}^n_+}\xi_i\frac{x_i}{|x|}d\sigma-\int_{B_\delta\cap\partial\mathbb{R}^n_+}\xi_n d\sigma\\
&\le \int_{B_\delta\cap\partial\mathbb{R}^n_+}(\frac{n+2}{2(n-2)}v_\epsilon\partial_n v_\epsilon|S_{nn}|^2-\frac{4n(n-1)}{(n-2)^2}v_\epsilon^{-1}\partial_n v_\epsilon\psi^2)d\sigma\\
&\quad -2\theta\sum_{i,k=1}^n\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|^2\epsilon^{n-2}\int_{B_\delta\cap\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|+2-2n}dx\\
&\quad +C\sum_{i,k=1}^n\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|^2\delta^{2|\alpha|+2-n}\epsilon^{n-2}.
\end{aligned}$$

For J_3 and J_4, by (9), Proposition 3 and Cauchy inequality,

$$\begin{aligned}
J_3+J_4 &\le C\sum_{i,k=1}^n\sum_{2\le|\alpha|\le d}(|h_{ik,\alpha}|^2(\epsilon+|x|)^{2|\alpha|+4-2n}+|h_{ik,\alpha}|(\epsilon+|x|)^{|\alpha|+d+3-2n})\epsilon^{n-2}\\
&\quad +C(\epsilon+|x|)^{2d+4-2n}\epsilon^{n-2}\\
&\le \theta\sum_{i,k=1}^n\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|^2\epsilon^{n-2}(\epsilon+|x|)^{2|\alpha|+2-2n}+C(\epsilon+|x|)^{2d+4-2n}\epsilon^{n-2}.
\end{aligned}$$

Thus,

$$\int_{B_\delta\cap\mathbb{R}^n_+}(J_3+J_4)dx\le\theta\sum_{i,k=1}^n\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|^2\epsilon^{n-2}\int_{B_\delta\cap\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|+2-2n}dx+C\delta^{2d+4-n}\epsilon^{n-2}.$$

Finally, by (3) we compute

$$\int_{B_\delta\cap\mathbb{R}^n_+}\frac{4(n-1)}{n-2}|dv_\epsilon|^2dx=\frac{4(n-1)}{n-2}(\int_{B_\delta\cap\partial\mathbb{R}^n_+}-v_\epsilon\partial_n v_\epsilon d\sigma+\int_{\partial B_\delta\cap\mathbb{R}^n_+}\sum_{i=1}^n v_\epsilon\partial_i v_\epsilon\frac{x_i}{|x|}d\sigma).$$

Combining the above, we obtain

$$\begin{aligned}
&\int_{B_\delta\cap\mathbb{R}^n_+}(\frac{4(n-1)}{n-2}|d(v_\epsilon+\psi)|_g^2+R_g(v_\epsilon+\psi)^2)dx\\
\leq & -\frac{4(n-1)}{n-2}\int_{B_\delta\cap\partial\mathbb{R}^n_+}(v_\epsilon\partial_n v_\epsilon+2\partial_n v_\epsilon\psi+\frac{n}{(n-2)}v_\epsilon^{-1}\partial_n v_\epsilon\psi^2)d\sigma\\
&+\frac{n+2}{2(n-1)}\int_{B_\delta\cap\partial\mathbb{R}^n_+}v_\epsilon\partial_n v_\epsilon|S_{nn}|^2d\sigma\\
&+\int_{\partial B_\delta\cap\mathbb{R}^n_+}\sum_{i=1}^n(\frac{4(n-1)}{n-2}v_\epsilon\partial_i v_\epsilon+v_\epsilon^2\partial_k h_{ik}-\partial_k v_\epsilon^2 h_{ik})\frac{x_i}{|x|}d\sigma\\
&-\theta\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|^2\epsilon^{n-2}\int_{B_\delta\cap\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|+2-2n}dx\\
&+C\epsilon^{n-2}\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|\delta^{-n+2+|\alpha|}+C\epsilon^{n-2}\delta^{2d+4-n}.
\end{aligned}$$

Finally, by (5) and $v_\epsilon\partial_n v_\epsilon|S_{nn}|^2\leq 0$ for $x\in\partial\mathbb{R}^n_+$,

$$\begin{aligned}
&-\frac{4(n-1)}{n-2}\int_{B_\delta\cap\partial\mathbb{R}^n_+}(v_\epsilon\partial_n v_\epsilon+2\partial_n v_\epsilon\psi+\frac{n}{(n-2)}v_\epsilon^{-1}\partial_n v_\epsilon\psi^2)d\sigma\\
&+\frac{n+2}{2(n-1)}\int_{B_\delta\cap\partial\mathbb{R}^n_+}v_\epsilon\partial_n v_\epsilon|S_{nn}|^2d\sigma\\
&\leq-\frac{4(n-1)}{n-2}\int_{B_\delta\cap\partial\mathbb{R}^n_+}(v_\epsilon\partial_n v_\epsilon+2\partial_n v_\epsilon\psi+\frac{n}{(n-2)}v_\epsilon^{-1}\partial_n v_\epsilon\psi^2)d\sigma\\
&+\frac{1}{2(n-1)}\int_{B_\delta\cap\partial\mathbb{R}^n_+}v_\epsilon\partial_n v_\epsilon|S_{nn}|^2d\sigma\\
&=4(n-1)\int_{B_\delta\cap\partial\mathbb{R}^n_+}v_\epsilon^{\frac{2}{n-2}}(v_\epsilon^2+2v_\epsilon\psi+\frac{n}{n-2}\psi^2-\frac{n-2}{8(n-1)^2}v_\epsilon^2|S_{nn}|^2)d\sigma.
\end{aligned}$$

This completes the proof. □

Proposition 8.

$$\begin{aligned}
&4(n-1)\int_{B_\delta\cap\partial\mathbb{R}^n_+}v_\epsilon^{\frac{2}{n-2}}(v_\epsilon^2+2v_\epsilon\psi+\frac{n}{n-2}\psi^2-\frac{n-2}{8(n-1)^2}v_\epsilon^2S_{nn}^2)d\sigma\\
&\leq\mathcal{Q}(B,\partial B)(\int_{B_\delta\cap\partial\mathbb{R}^n_+}(v_\epsilon+\psi)^{\frac{2(n-1)}{n-2}}d\sigma)^{\frac{n-2}{n-1}}+C\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|\delta^{|\alpha|-n+1}\epsilon^{n-1}\\
&+C\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|^2\epsilon^{n-1}\delta^2\int_{B_\delta\cap\partial\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|-2n+2}d\sigma
\end{aligned}$$

for $0<2\epsilon\leq\delta\leq\delta_0$ and δ_0 sufficiently small.

Proof. Recall that

$$\mathcal{Q}(B^n,\partial B^n)\left(\int_{\partial\mathbb{R}^n_+} v_\epsilon^{\frac{2(n-1)}{n-2}}d\sigma\right)^{\frac{n-2}{n-1}}=\frac{4(n-1)}{n-2}\int_{\mathbb{R}^n_+}|\nabla v_\epsilon|^2dx$$

and

$$\int_{\mathbb{R}^n_+}|\nabla v_\epsilon|^2dx=(n-2)\int_{\partial\mathbb{R}^n_+} v_\epsilon^{\frac{2(n-1)}{n-2}}d\sigma.$$

Then it follows that

$$4(n-1)(\int_{\partial\mathbb{R}^n_+} v_\epsilon^{\frac{2(n-1)}{n-2}}d\sigma)^{\frac{1}{n-1}}=\mathcal{Q}(B,\partial B).$$

Besides, since $V_n=0$ on $\partial\mathbb{R}^n_+$, we have

$$\begin{aligned}\psi&=\frac{n-2}{2(n-1)}v_\epsilon^{-\frac{n}{n-2}}\sum_{a=1}^{n-1}\partial_a(v_\epsilon^{\frac{2(n-1)}{n-2}}V_a)+\frac{n-2}{4(n-1)}v_\epsilon(2\partial_nV_n-\frac{2}{n}divV)\\&=\frac{n-2}{2(n-1)}v_\epsilon^{-\frac{n}{n-2}}\sum_{a=1}^{n-1}\partial_a(v_\epsilon^{\frac{2(n-1)}{n-2}}V_a)+\frac{n-2}{4(n-1)}v_\epsilon S_{nn}\end{aligned}$$

for $x\in\partial\mathbb{R}^n_+$. Moreover, by (9)

$$\int_{\partial B_\delta\cap\partial\mathbb{R}^n_+} v_\epsilon^{\frac{2(n-1)}{n-2}}\sum_{a=1}^{n-1}V_a\frac{x_a}{|x|}d\mu\le C\sum_{i,k=1}^n\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|\delta^{|\alpha|-n+1}\epsilon^{n-1}.$$

Thus,

$$\int_{B_\delta\cap\partial\mathbb{R}^n_+}2v_\epsilon^{\frac{n}{n-2}}\psi d\sigma-\int_{B_\delta\cap\partial\mathbb{R}^n_+}\frac{n-2}{2(n-1)}v_\epsilon^{\frac{2(n-1)}{n-2}}S_{nn}d\sigma\le C\sum_{i,k=1}^n\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|\delta^{|\alpha|-n+1}\epsilon^{n-1}.$$

Putting above together and using Holder inequality, we get

$$\begin{aligned}&4(n-1)\int_{B_\delta\cap\partial\mathbb{R}^n_+}v_\epsilon^{\frac{2}{n-2}}(v_\epsilon^2+2v_\epsilon\psi+\frac{n}{n-2}\psi^2-\frac{n-2}{8(n-1)^2}v_\epsilon^2S_{nn}^2)d\sigma\\ \le\ &4(n-1)\int_{B_\delta\cap\partial\mathbb{R}^n_+}v_\epsilon^{\frac{2}{n-2}}(v_\epsilon^2+\frac{n-2}{2(n-1)}v_\epsilon^2S_{nn}+\frac{n}{n-2}\psi^2-\frac{n-2}{8(n-1)^2}v_\epsilon^2S_{nn}^2)d\sigma\\ &+C\sum_{i,k=1}^n\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|\delta^{|\alpha|-n+1}\epsilon^{n-1}\\ \le\ &\mathcal{Q}(B,\partial B)\left(\int_{B_\delta\cap\partial\mathbb{R}^n_+}(v_\epsilon^2+\frac{n-2}{2(n-1)}v_\epsilon^2S_{nn}+\frac{n}{n-2}\psi^2-\frac{n-2}{8(n-1)^2}v_\epsilon^2S_{nn}^2)^{\frac{n-1}{n-2}}d\sigma\right)^{\frac{n-2}{n-1}}\\ &+C\sum_{i,k=1}^n\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|\delta^{|\alpha|-n+1}\epsilon^{n-1}.\end{aligned}$$

We next notice that by Taylor expansion, there exists a constant $C_0 = C_0(n)$ such that

$$
\begin{aligned}
&(1+\frac{n-2}{2(n-1)}y+\frac{n}{n-2}z^2-\frac{n-2}{8(n-1)^2}y^2)^{\frac{n-1}{n-2}}-(1+z)^{\frac{2(n-1)}{n-2}}+\frac{2(n-1)}{n-2}z-\frac{1}{2}y \\
&\leq C_0(|y|^3+|z|^3)
\end{aligned}
$$

for $|y|,|z| \leq \frac{1}{2}$. By (9), $|S_{nn}| \leq \frac{1}{2}$ and $|\psi| \leq \frac{1}{2}v_\epsilon$ for $|x| \leq \delta$. Hence,

$$
\begin{aligned}
&(v_\epsilon^2+\frac{n-2}{2(n-1)}v_\epsilon^2 S_{nn}+\frac{n}{n-2}\psi^2-\frac{n-2}{8(n-1)^2}v_\epsilon^2 S_{nn}^2)^{\frac{n-1}{n-2}}-(v_\epsilon+\psi)^{\frac{2(n-1)}{n-2}} \\
&\quad+\frac{2(n-1)}{n-2}v_\epsilon^{\frac{n}{n-2}}\psi-\frac{1}{2}v_\epsilon^{\frac{2(n-1)}{n-2}}S_{nn} \\
&\leq C_0 v_\epsilon^{\frac{2(n-1)}{n-2}}(|\frac{\psi}{v_\epsilon}|^3+|S_{nn}|^3) \leq C v_\epsilon^{\frac{2(n-1)}{n-2}}(|\frac{\psi}{v_\epsilon}|^2+|S_{nn}|^2)\delta^2 \\
&\leq C\sum_{i,k=1}^{n}\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|^2(\epsilon+|x|)^{2|\alpha|-2n+2}\epsilon^{n-1}\delta^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_{B_\delta\cap\partial\mathbb{R}^n_+}(v_\epsilon^2+\frac{n-2}{2(n-1)}v_\epsilon^2 S_{nn}+\frac{n}{n-2}\psi^2-\frac{n-2}{8(n-1)^2}v_\epsilon^2 S_{nn}^2)^{\frac{n-1}{n-2}}d\sigma \\
\leq\ &\int_{B_\delta\cap\partial\mathbb{R}^n_+}(v_\epsilon+\psi)^{\frac{2(n-1)}{n-2}}d\sigma+\int_{B_\delta\cap\partial\mathbb{R}^n_+}\frac{2(n-1)}{n-2}v_\epsilon^{\frac{n}{n-2}}\psi d\sigma-\int_{B_\delta\cap\partial\mathbb{R}^n_+}\frac{1}{2}v_\epsilon^{\frac{2(n-1)}{n-2}}S_{nn}d\sigma \\
+\ &C\sum_{i,k=1}^{n}\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|^2\epsilon^{n-1}\delta^2\int_{B_\delta\cap\partial\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|-2n+2}d\sigma \\
\leq\ &\int_{B_\delta\cap\partial\mathbb{R}^n_+}(v_\epsilon+\psi)^{\frac{2(n-1)}{n-2}}d\sigma+C\sum_{i,k=1}^{n}\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|^2\epsilon^{n-1}\delta^2\int_{B_\delta\cap\partial\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|-2n+2}d\sigma \\
+\ &C\sum_{i,k=1}^{n}\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|\delta^{|\alpha|-n+1}\epsilon^{n-1}.
\end{aligned}
$$

This completes the proof. □

4 Proof of the main theorems

In this section, we construct a test function $\phi_{(\epsilon,\delta)}$ with energy functional less than $\mathcal{Q}(B,\partial B)$ and prove Theorem 2 and 3. Since the case that $\mathcal{Q}(M,\partial M,g)\leq 0$ is trivial, it suffices to consider $\mathcal{Q}(M,\partial M,g)>0$.

After a conformal change of the metric, we may assume ∂M is totally geodesic. Let $p\in\partial M$ and let $(x_1,\cdots,x_n)$ be the conformal Fermi coordinates around p described in

Section 2. We denote by G the Green's function of the conformal Laplacian with pole at p which satisfies the Neumann boundary condition. We assume that G is normalized such that $\lim_{|x|\to 0}|x|^{n-2}G(x)=1$. Then G satisfies [6]

$$|G(x)-|x|^{2-n}|\le C\sum_{i,k=1}^{n}\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}||x|^{|\alpha|+2-n}+C|x|^{d+3-n}. \tag{12}$$

Moreover, we define as in [6] a flux integral

$$\begin{aligned}\mathcal{I}(p,\delta) &= \frac{4(n-1)}{n-2}\int_{\partial B_\delta\cap\mathbb{R}^n_+}\sum_{i=1}^{n}(|x|^{2-n}\partial_i G-G\partial_i|x|^{2-n})\frac{x_i}{|x|}d\sigma\\ &\quad-\int_{\partial B_\delta\cap\mathbb{R}^n_+}\sum_{i,k=1}^{n}|x|^{2-2n}(|x|^2\partial_k h_{ik}-2nx_kh_{ik})\frac{x_i}{|x|}d\sigma\end{aligned}$$

for $\delta>0$ sufficiently small.

We define

$$\phi_{(\epsilon,\delta)}=\eta_\delta(v_\epsilon+\psi)+(1-\eta_\delta)\epsilon^{\frac{n-2}{2}}G,$$

where ψ is the function constructed in Section 2. We recall that

$$\epsilon^{\frac{n-2}{2}}(\epsilon+|x|)^{-n+2}\le v_\epsilon(x)\le C(n)\epsilon^{\frac{n-2}{2}}(\epsilon+|x|)^{-n+2}\quad\text{for } x\in\mathbb{R}^n_+;$$
$$|\partial v_\epsilon|(x)\le C(n)\epsilon^{\frac{n-2}{2}}(\epsilon+|x|)^{-n+1}\quad\text{for } x\in\mathbb{R}^n_+;$$

and

$$|v_\epsilon-\epsilon^{\frac{n-2}{2}}|x|^{-n+2}|\le C(n)\epsilon^{\frac{n}{2}}|x|^{-n+1}\quad\text{for } x\in\mathbb{R}^n_+,\ \text{and } |x|\ge 2\epsilon. \tag{13}$$

Proposition 9.

$$\begin{aligned}&\int_M(\frac{4(n-1)}{n-2}|d\phi_{(\epsilon,\delta)}|_g^2+R_g\phi_{(\epsilon,\delta)}^2)dV_g\\ &\le\mathcal{Q}(B,\partial B)(\int_{\partial M}\phi_{(\epsilon,\delta)}^{\frac{2(n-1)}{n-2}}d\sigma_g)^{\frac{n-2}{n-1}}-\frac{\theta}{2}\sum_{i,k=1}^{n}\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|^2\epsilon^{n-2}\int_{B_\delta\cap\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|+2-2n}dx\\ &-\epsilon^{n-2}\mathcal{I}(p,\delta)+C\epsilon^{n-2}\sum_{i,k=1}^{n}\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|\delta^{-n+2+|\alpha|}+C\epsilon^{n-2}\delta^{2d+4-n}+C\delta^{-n+1}\epsilon^{n-1}\end{aligned}$$

for $0<2\epsilon\le\delta\le\delta_0$ and δ_0 sufficiently small.

Proof. Let Ω_δ be the coordinates ball of radius δ in Fermi coordinates. In other words, $(x_1,\cdots,x_n)$ satisfies $x_1^2+\cdots+x_n^2<\delta^2$ and $x_n\ge 0$. By divergence theorem

$$\begin{aligned}&\int_{M\setminus\Omega_\delta}(\frac{4(n-1)}{n-2}|\nabla_g\phi_{(\epsilon,\delta)}|^2+R_g\phi_{(\epsilon,\delta)}^2)dV_g\\ &=\int_{M\setminus\Omega_\delta}-(\frac{4(n-1)}{n-2}\Delta_g\phi_{(\epsilon,\delta)}-R_g\phi_{(\epsilon,\delta)})(\phi_{(\epsilon,\delta)}-\epsilon^{\frac{n-2}{2}}G)dV_g\\ &\quad+\frac{4(n-1)}{n-2}\int_{\partial(M\setminus\Omega_\delta)}(\nabla_{\nu_g}\phi_{(\epsilon,\delta)}\phi_{(\epsilon,\delta)}+\epsilon^{\frac{n-2}{2}}(\phi_{(\epsilon,\delta)}\nabla_{\nu_g}G-G\nabla_{\nu_g}\phi_{(\epsilon,\delta)}))d\sigma_g,\end{aligned}$$

where ν_g is the unit outer normal on $\partial(M \setminus \Omega_\delta)$ with respect to g. Notice that

$$\partial(M \setminus \Omega_\delta) = (\partial M \setminus \Omega_\delta) \cup (\partial\Omega_\delta \setminus \partial M).$$

We will compute the above integral in several steps.

We first notice that for $x \in M \setminus \Omega_\delta$, we have $\phi_{(\epsilon,\delta)} - \epsilon^{\frac{n-2}{2}} G = \eta_\delta(v_\epsilon + \psi - \epsilon^{\frac{n-2}{2}} G)$. In particular, $\phi_{(\epsilon,\delta)} - \epsilon^{\frac{n-2}{2}} G = 0$ in $M \setminus \Omega_{2\delta}$. By (12) and (9),

$$\begin{aligned}
&\sup_{M\setminus\Omega_\delta} (|\phi_{(\epsilon,\delta)} - \epsilon^{\frac{n-2}{2}} G| + \delta^2 |\frac{4(n-1)}{n-2}\Delta_g \phi_{(\epsilon,\delta)} - R_g \phi_{(\epsilon,\delta)}|) \\
&\leq C \sum_{i,k=1}^{n} \sum_{2\leq|\alpha|\leq d} |h_{ik,\alpha}| \delta^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C\delta^{d+3-n}\epsilon^{\frac{n-2}{2}} + C\delta^{-n+1}\epsilon^{\frac{n}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&- \int_{M\setminus\Omega_\delta} (\frac{4(n-1)}{n-2}\Delta_g \phi_{(\epsilon,\delta)} - R_g\phi_{(\epsilon,\delta)})(\phi_{(\epsilon,\delta)} - \epsilon^{\frac{n-2}{2}} G) dV_g \\
\leq\ & C \sum_{i,k=1}^{n} \sum_{2\leq|\alpha|\leq d} |h_{ik,\alpha}|^2 \delta^{2|\alpha|+2-n} \epsilon^{n-2} + C\delta^{2d+4-n}\epsilon^{n-2} + C\delta^{-n}\epsilon^{n}.
\end{aligned}$$

We now compute the boundary terms on $\partial M \setminus \Omega_\delta$. Since $\nabla_{\nu_g} G = 0$ on ∂M, by (5), Proposition 5 and (9)

$$\sup_{\partial M\cap(\Omega_{2\delta}\setminus\Omega_\delta)} |\nabla_{\nu_g}\phi_{(\epsilon,\delta)}| \leq \sup_{\partial M\cap(\Omega_{2\delta}\setminus\Omega_\delta)} |\partial_n v_\epsilon + \partial_n \psi| \leq C\epsilon^{\frac{n}{2}}\delta^{-n} + C\sum_{i,k=1}^{n} \sum_{2\leq|\alpha|\leq d} |h_{ik,\alpha}| \delta^{|\alpha|-n}\epsilon^{\frac{n}{2}}.$$

Hence,

$$\begin{aligned}
&\int_{\partial M\setminus\Omega_\delta} (\nabla_{\nu_g}\phi_{(\epsilon,\delta)}\phi_{(\epsilon,\delta)} + \epsilon^{\frac{n-2}{2}}(\phi_{(\epsilon,\delta)}\nabla_{\nu_g} G - G\nabla_{\nu_g}\phi_{(\epsilon,\delta)})) d\sigma_g \\
&= \int_{\partial M\setminus\Omega_\delta} \nabla_{\nu_g}\phi_{(\epsilon,\delta)}(\phi_{(\epsilon,\delta)} - \epsilon^{\frac{n-2}{2}} G) d\sigma_g \\
&\leq C \sum_{i,k=1}^{n} \sum_{2\leq|\alpha|\leq d} |h_{ik,\alpha}| \delta^{|\alpha|+1-n}\epsilon^{n-1} + C\delta^{2d+4-n}\epsilon^{n-2} + C\delta^{-n}\epsilon^{n}.
\end{aligned}$$

We next compute the boundary terms on $\partial\Omega_\delta \setminus \partial M$.

$$\begin{aligned}
\int_{\partial\Omega_\delta\setminus\partial M} \nabla_{\nu_g}\phi_{(\epsilon,\delta)}\phi_{(\epsilon,\delta)} d\sigma_g \ &\leq\ \int_{\partial B_\delta\cap\mathbb{R}^n_+} \sum_{i=1}^{n} (-\partial_i v_\epsilon v_\epsilon + \sum_{k=1}^{n} v_\epsilon \partial_k v_\epsilon h_{ik}) \frac{x_i}{|x|} d\sigma \\
&+\ C \sum_{i,k=1}^{n} \sum_{2\leq|\alpha|\leq d} |h_{ik,\alpha}| \delta^{|\alpha|+2-n}\epsilon^{n-2} + C\delta^{2d+4-n}\epsilon^{n-2}.
\end{aligned}$$

Also,

$$\int_{\partial\Omega_\delta\setminus\partial M}(\phi_{(\epsilon,\delta)}\nabla_{\nu_g}G - G\nabla_{\nu_g}\phi_{(\epsilon,\delta)})d\sigma_g \leq -\int_{\partial B_\delta\cap\mathbb{R}^n_+}\sum_{i=1}^n(v_\epsilon\partial_i G - G\partial_i v_\epsilon)\frac{x_i}{|x|}d\sigma$$
$$+ C\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|\delta^{|\alpha|+2-n}\epsilon^{\frac{n-2}{2}} + C\delta^{2d+4-n}\epsilon^{\frac{n-2}{2}}.$$

Combining the above, we obtain

$$\begin{aligned}
&\int_{M\setminus\Omega_\delta}(\frac{4(n-1)}{n-2}|d\phi_{(\epsilon,\delta)}|_g^2 + R_g\phi^2_{(\epsilon,\delta)})dV_g\\
\leq\ & -\frac{4(n-1)}{n-2}\int_{\partial B_\delta\cap\mathbb{R}^n_+}\sum_{i=1}^n(\partial_i v_\epsilon v_\epsilon - \sum_{k=1}^n v_\epsilon\partial_k v_\epsilon h_{ik} + \epsilon^{\frac{n-2}{2}}(v_\epsilon\partial_i G - G\partial_i v_\epsilon))\frac{x_i}{|x|}d\sigma\\
+\ & C\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|\delta^{|\alpha|+2-n}\epsilon^{n-2} + C\delta^{2d+4-n}\epsilon^{n-2} + C\delta^{-n}\epsilon^n.
\end{aligned}$$

On the other hand, by Proposition 7 and 8,

$$\begin{aligned}
&\int_{\Omega_\delta}(\frac{4(n-1)}{n-2}|d\phi_{(\epsilon,\delta)}|_g^2 + R_g(\phi_{(\epsilon,\delta)})^2)dV_g\\
\leq&\mathcal{Q}(B,\partial B)(\int_{\partial M}\phi_{(\epsilon,\delta)}^{\frac{2(n-1)}{n-2}}d\sigma_g)^{\frac{n-2}{n-1}} + \int_{\partial B_\delta\cap\mathbb{R}^n_+}\sum_{i=1}^n(\frac{4(n-1)}{n-2}v_\epsilon\partial_i v_\epsilon + v_\epsilon^2\partial_k h_{ik} - \partial_k v_\epsilon^2 h_{ik})\frac{x_i}{|x|}d\sigma\\
&-\ \theta\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|^2\epsilon^{n-2}\int_{B_\delta\cap\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|+2-2n}dx\\
&+C\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|^2\epsilon^{n-1}\delta^2\int_{B_\delta\cap\partial\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|-2n+2}d\sigma\\
&+\ C\sum_{i,k=1}^n\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|\delta^{-n+2+|\alpha|}\epsilon^{n-2} + C\delta^{2d+4-n}\epsilon^{n-2}.
\end{aligned}$$

Adding the above two inequalities, we get

$$
\begin{aligned}
&\int_M \left(\frac{4(n-1)}{n-2}|d\phi_{(\epsilon,\delta)}|_g^2 + R_g(\phi_{(\epsilon,\delta)})^2\right)dV_g\\
\le &\mathcal{Q}(B,\partial B)\left(\int_{\partial M} \phi_{(\epsilon,\delta)}^{\frac{2(n-1)}{n-2}} d\sigma_g\right)^{\frac{n-2}{n-1}} + \int_{\partial B_\delta \cap \mathbb{R}^n_+} \sum_{i=1}^n \left(v_\epsilon^2 \partial_k h_{ik} + \frac{n}{n-2}\partial_k v_\epsilon^2 h_{ik}\right)\frac{x_i}{|x|}d\sigma\\
&- \frac{4(n-1)}{n-2}\int_{\partial B_\delta \cap \mathbb{R}^n_+} \sum_{i=1}^n \epsilon^{\frac{n-2}{2}}(v_\epsilon \partial_i G - G\partial_i v_\epsilon)\frac{x_i}{|x|}d\sigma\\
&- \ \theta \sum_{i,k=1}^n \sum_{2\le|\alpha|\le d} |h_{ik,\alpha}|^2 \epsilon^{n-2}\int_{B_\delta\cap\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|+2-2n}dx\\
&+ C\sum_{i,k=1}^n \sum_{2\le|\alpha|\le d} |h_{ik,\alpha}|^2 \epsilon^{n-1}\delta^2 \int_{B_\delta\cap\partial\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|-2n+2}d\sigma\\
&+ \ C\sum_{i,k=1}^n \sum_{2\le|\alpha|\le d} |h_{ik,\alpha}|\delta^{-n+2+|\alpha|}\epsilon^{n-2} + C\delta^{2d+4-n}\epsilon^{n-2} + C\delta^{-n}\epsilon^n.
\end{aligned}
$$

Since

$$
\epsilon^{n-1}\delta^2 \int_{B_\delta\cap\partial\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|-2n+2}d\sigma = \epsilon^{2|\alpha|}\delta^2\int_0^{\frac{\delta}{\epsilon}}(1+t)^{2|\alpha|-2n+2}t^{n-2}dt
$$

and

$$
\epsilon^{n-2}\int_{B_\delta\cap\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|+2-2n}dx = \epsilon^{2|\alpha|}\int_0^{\frac{\delta}{\epsilon}}(1+t)^{2|\alpha|-2n+2}t^{n-1}dt,
$$

then for δ sufficiently small and $2\epsilon \le \delta$, we have

$$
C\epsilon^{n-1}\delta^2\int_{B_\delta\cap\partial\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|-2n+2}d\sigma < \frac{\theta}{2}\,\epsilon^{n-2}\int_{B_\delta\cap\mathbb{R}^n_+}(\epsilon+|x|)^{2|\alpha|+2-2n}dx.
$$

Moreover, by (12) and (13)

$$
\begin{aligned}
&\int_{\partial B_\delta\cap\mathbb{R}^n_+}\left(\sum_{i=1}^n\left(v_\epsilon^2\partial_k h_{ik} + \frac{n}{n-2}\partial_k v_\epsilon^2 h_{ik}\right) - \frac{4(n-1)}{n-2}\sum_{i=1}^n \epsilon^{\frac{n-2}{2}}(v_\epsilon\partial_i G - G\partial_i v_\epsilon)\right)\frac{x_i}{|x|}d\sigma\\
\le \ &-\epsilon^{n-2}\mathcal{I}(p,\delta) + C\sum_{i,k=1}^n\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}|\delta^{|\alpha|-n+2}\epsilon^{n-2} + C\epsilon^{n-1}\delta^{-n+1}.
\end{aligned}
$$

From these the assertion follows. □

We are ready to prove Theorem 2.

Proof of Theorem 2. Since $p \notin \mathcal{Z}$, we have $\sum_{i,k=1}^{n}\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}| > 0$. Thus, by Proposition 9,

$$\int_M (\frac{4(n-1)}{n-2}|d\phi_{(\epsilon,\delta)}|_g^2 + R_g\phi_{(\epsilon,\delta)}^2)dV_g < \mathcal{Q}(B,\partial B)(\int_{\partial M}\phi_{(\epsilon,\delta)}^{\frac{2(n-1)}{n-2}}d\sigma_g)^{\frac{n-2}{n-1}}$$

for $\epsilon > 0$ sufficiently small. This completes the proof. □

Now we consider the case that $p \in \mathcal{Z}$. We recall a result about $\mathcal{I}(p,\delta)$.

Proposition 10. *[6] Let $p \in \partial M$. Suppose $p \in \mathcal{Z}$.*

(i) The limit $\lim_{\delta\to 0}\mathcal{I}(p,\delta)$ exists.

(ii)The doubling of $(M\backslash\{p\}, G^{\frac{4}{n-2}}g)$ has a well-defined mass which equals $\lim_{\delta\to 0}\mathcal{I}(p,\delta)$ up to a positive factor.

Proof of Theorem 3. Since $p \in \mathcal{Z}$, we have $\sum_{i,k=1}^{n}\sum_{2\le|\alpha|\le d}|h_{ik,\alpha}| = 0$. By Proposition 9,

$$\begin{aligned}\int_M (\frac{4(n-1)}{n-2}|d\phi_{(\epsilon,\delta)}|_g^2 + R_g\phi_{(\epsilon,\delta)}^2)dV_g &\le \mathcal{Q}(B,\partial B)(\int_{\partial M}\phi_{(\epsilon,\delta)}^{\frac{2(n-1)}{n-2}}d\sigma_g)^{\frac{n-2}{n-1}} \\ &\quad - \epsilon^{n-2}\mathcal{I}(p,\delta) + C\delta^{2d+4-n}\epsilon^{n-2} + C\delta^{-n+1}\epsilon^{n-1}\end{aligned}$$

for $0 < 2\epsilon \le \delta$. By assumption $\lim_{\delta\to 0}\mathcal{I}(p,\delta) > 0$, we may choose δ sufficiently small such that $\mathcal{I}(p,\delta) - C\delta^{2d+4-n} > 0$. We next choose $0 < \epsilon < \frac{\delta}{2}$ sufficiently small such that $\mathcal{I}(p,\delta) - C\delta^{2d+4-n} - C\delta^{-n+1}\epsilon > 0$. Then

$$\int_M (\frac{4(n-1)}{n-2}|d\phi_{(\epsilon,\delta)}|_g^2 + R_g\phi_{(\epsilon,\delta)}^2)dV_g < \mathcal{Q}(B,\partial B)(\int_{\partial M}\phi_{(\epsilon,\delta)}^{\frac{2(n-1)}{n-2}}d\sigma_g)^{\frac{n-2}{n-1}}.$$

□

Appendix: An elliptic system in $\mathbb{R}^n_+$

In the appendix, we solve a boundary value problem for an elliptic system in $\mathbb{R}^n_+$.

Let $B_{\frac{1}{2}}$ be the ball of radius $\frac{1}{2}$ equipped with the flat metric g. We denote by $\mathcal{X}$ the space of vector fields $V \in H^1(B_{\frac{1}{2}})$ such that $\langle V,\nu\rangle = 0$ on $\partial B_{\frac{1}{2}}$, where ν is the unit outer normal on $\partial B_{\frac{1}{2}}$. We also denote by $\mathcal{Y}$ the space of trace-free symmetric two-tensors on $B_{\frac{1}{2}}$ of class L^2. Let $\mathcal{D}: \mathcal{X}\to\mathcal{Y}$ be the conformal killing operator, which satisfies

$$(\mathcal{D}V)_{ik} = V_{i,k} + V_{k,i} - \frac{2}{n}divV g_{ik}.$$

By stereographic projection, $B_{\frac{1}{2}}$ is conformal to the hemisphere $\mathbb{S}^n_+$ with standard metric g_c. The metric g_c satisfies $g_c = u^{\frac{4}{n-2}}g$, where $u = (\frac{2}{1+4|x|^2})^{\frac{n-2}{2}}$ for $|x| \le \frac{1}{2}$. We may define similarly $\mathcal{X}^*$ the space of vector fields $V \in H^1(\mathbb{S}^n_+)$ such that $\langle V,\nu\rangle = 0$ on $\partial\mathbb{S}^n_+$, where ν is the unit outer normal on $\partial\mathbb{S}^n_+$, $\mathcal{Y}^*$ the space of trace-free symmetric two-tensors on $\mathbb{S}^n_+$ of class L^2 and $\mathcal{D}^*: \mathcal{X}^*\to\mathcal{Y}^*$ the conformal killing operator on the hemisphere. Then it follows that $V \in H^1(\mathbb{S}^n_+)$ if and only if $V \in H^1(B_{\frac{1}{2}})$, and $\mathcal{D}^*V = 0$ if and only if $\mathcal{D}V = 0$.

Lemma 1. $\ker \mathcal{D}$ *is finite dimensional.*

Proof. In [6], it was shown (after Lemma 21) that $\ker \mathcal{D}^*$ is finite dimensional. Then the assertion follows easily. $\square$

We now define $\mathcal{X}_0 = \{V \in \mathcal{X} : \langle V, U\rangle_{L^2(B_{\frac{1}{2}})} = 0 \text{ for all } U \in \ker \mathcal{D}\}$.

Lemma 2. *For all $V \in \mathcal{X}_0$, it holds $\|V\|^2_{H^1(B_{\frac{1}{2}})} \leq C\|\mathcal{D}V\|^2_{L^2(B_{\frac{1}{2}})}$, where $C = C(n)$.*

Proof. Suppose the inequality does not hold, then there exist a sequence of vector fields $V^{(j)} \in \mathcal{X}_0$ such that $\|V^{(j)}\|_{H^1(B_{\frac{1}{2}})} = 1$ for all j and $\|\mathcal{D}V^{(j)}\|_{L^2(B_{\frac{1}{2}})} \to 0$ as $j \to \infty$. By passing to a subsequence, $V^{(j)} \rightharpoonup V^{(0)}$ weakly in $H^1(B_{\frac{1}{2}})$ for some $V^{(0)} \in \mathcal{X}_0$. It follows that $\mathcal{D}V^{(0)} = 0$, and as a result $V^{(0)} = 0$. Notice that $V^{(j)} \to V^{(0)}$ strongly in $L^2(B_{\frac{1}{2}})$. Thus, $\|V^{(j)}\|_{L^2(B_{\frac{1}{2}})} \to 0$. Therefore, $\|V^{(j)}\|_{L^2(\mathbb{S}^n_+)} \to 0$. By [6] Lemma 21, $\|V^{(j)}\|_{H^1(\mathbb{S}^n_+)} \to 0$ as $j \to \infty$. Hence, $\|V^{(j)}\|_{H^1(B_{\frac{1}{2}})} \to 0$ as $j \to \infty$. This gives a contradiction. $\square$

Proposition 11. *Let h be a two-tensor in $\mathcal{Y}$. Then there exists a unique vector field $V \in \mathcal{X}_0$ such that $\langle h - \mathcal{D}V, \mathcal{D}U\rangle_{L^2(B_{\frac{1}{2}})} = 0$ for all $U \in \mathcal{X}$.*

Moreover, $\|V\|^2_{H^1(B_{\frac{1}{2}})} \leq C\|h\|^2_{L^2(B_{\frac{1}{2}})}$, where $C = C(n)$.

Proof. It follows by the same argument in [6] Proposition 23, and Lemma 2 above that the minimizer of $\|h - \mathcal{D}V\|^2_{L^2(B_{\frac{1}{2}})}$ exists in $\mathcal{X}_0$, which satisfies the required properties. $\square$

We now consider another conformal map. The ball $B_{\frac{1}{2}}$ is conformal to $\mathbb{R}^n_+ \cup \{\infty\}$. The metric g satisfies $g = v^{\frac{4}{n-2}}\delta$, where

$$v = \left(\frac{1}{(1+x_n)^2 + \sum_{1\leq a\leq n-1} x_a^2)}\right)^{\frac{n-2}{2}}.$$

Proposition 12. *Let h be a smooth trace-free symmetric two-tensor on $\mathbb{R}^n_+$ with compact support. Then there exists a smooth vector field V on $\mathbb{R}^n_+$ such that*

$$\begin{cases} \sum_{k=1}^n \partial_k[v^{\frac{2n}{n-2}}(h_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} divV \delta_{ik})] = 0 & in\, \mathbb{R}^n_+ \\ \partial_n V_a - h_{an} = 0 & on\, \partial\mathbb{R}^n_+ \\ V_n = 0 & on\, \partial\mathbb{R}^n_+ \end{cases}$$

for $i = 1, \cdots, n$ and $a = 1, \cdots, n-1$. Moreover,

$$\int_{\mathbb{R}^n_+} v^{\frac{2(n+2)}{n-2}}|V|^2 dx \leq C\int_{\mathbb{R}^n_+} v^{\frac{2n}{n-2}}|h|^2 dx,$$

where $C = C(n)$.

Proof. By Proposition 11, there exists a smooth vector field V such that

$$\int_{\mathbb{R}^n_+} v^{\frac{2(n+2)}{n-2}}(h_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} divV\delta_{ik})\partial_k U_i dx = 0$$

for all $U \in \mathcal{X}$ and $V_n = 0$ on $\partial\mathbb{R}^n_+$. By elliptic regularity ([14] pp.245-249), V is smooth. Hence, $\sum_{k=1}^n \partial_k[v^{\frac{2n}{n-2}}(h_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} divV\delta_{ik})] = 0$ on $\mathbb{R}^n_+$ and $\partial_n V_a - h_{an} = 0$ on $\partial\mathbb{R}^n_+$. □

Proposition 13. *Let $h_{ik} = \eta(\frac{|x|}{\rho})\sum_{|\alpha|=2}^d h_{ik,\alpha}x^\alpha$ be a trace-free symmetric two-tensor, where $d = [\frac{n-2}{2}]$, $\rho \geq 1$ and $\eta(t)$ be a fixed cut-off function which satisfies $\eta(t) = 0$ for $t \geq 2$. Suppose V is the vector field constructed in Proposition 12. Then for $x \in \mathbb{R}^n_+$,*

$$|\partial^\beta V|^2(x) \leq C(n, |\beta|)\sum_{i,k}\sum_{2\leq|\alpha|\leq d}|h_{ik,\alpha}|^2(1+|x|)^{2|\alpha|+2-2|\beta|}$$

for every multi-index β.

Proof. The proof is similar to [4] Proposition 23 and Corollary 24.

Without loss of generality we may assume $h_{ik} = \eta(\frac{|x|}{\rho})\sum_{|\alpha|=l} h_{ik,\alpha}x^\alpha$, where $2 \leq l \leq d$. We first prove that

$$\sup_{r\geq 1} r^{-2l-n-2}\int_{(B_{2r}\setminus B_r)\cap\mathbb{R}^n_+}|V|^2 dx \leq C\int_{\mathbb{R}^n_+}((1+x_n)^2 + \sum_{a=1}^{n-1}x_a^2)^{-n-2}|V|^2 dx$$
$$+ C\sup_{r\geq 1} r^{-2l-n}\int_{(B_{2r}\setminus B_r)\cap\mathbb{R}^n_+}|h|^2 dx. \tag{14}$$

Suppose (14) does not hold, there exist sequences $h_{ik}^{(s)}$ and $V^{(s)}$ such that

$$\sup_{r\geq 1} r^{-2l-n-2}\int_{(B_{2r}\setminus B_r)\cap\mathbb{R}^n_+}|V^{(s)}|^2 dx = 1,$$

$$\lim_{s\to\infty}\int_{\mathbb{R}^n_+}((1+x_n)^2 + \sum_{a=1}^{n-1}x_a^2)^{-(n+2)}|V^{(s)}|^2 dx = 0,$$

and

$$\lim_{s\to\infty}\sup_{r\geq 1} r^{-2l-n}\int_{(B_{2r}\setminus B_r)\cap\mathbb{R}^n_+}|h^{(s)}|^2 dx = 0.$$

Therefore, there exists a sequence $\rho^{(s)} \to \infty$ such that

$$(\rho^{(s)})^{-2l-n-2}\int_{(B_{2\rho^{(s)}}\setminus B_{\rho^{(s)}})\cap\mathbb{R}^n_+}|V^{(s)}|^2 dx \geq \frac{1}{2}.$$

Let $\tilde{h}^{(s)}_{ik} = (\rho^{(s)})^{-l}|x|^{-4}(|x|^2\delta_{ij}-2x_ix_j)(|x|^2\delta_{kl}-2x_kx_l)h^{(s)}_{jl}(\frac{\rho^{(s)}x}{|x|^2})$, and $\tilde{V}_j = (\rho^{(s)})^{-l-1}(|x|^2\delta_{ij}-2x_ix_j)V^{(s)}_i(\frac{\rho^{(s)}x}{|x|^2})$. Then they satisfy

$$\sum_{k=1}^n \partial_k[((1+\frac{x_n}{\rho^{(s)}})^2+\sum_{a=1}^{n-1}(\frac{x_a}{\rho^{(s)}})^2)^{-n}(\tilde{h}_{ik}-\partial_i\tilde{V}_k-\partial_k\tilde{V}_i+\frac{2}{n}div\tilde{V}\delta_{ik})]=0$$

in $\mathbb{R}^n_+$ for $i=1,\cdots,n$, and $\tilde{V}_n = \partial_n\tilde{V}_a - \tilde{h}_{an} = 0$ on $\partial\mathbb{R}^n_+$. Thus, by passing to a subsequence, $\tilde{V}_j$ converges weakly to a vector field $V \in W^{1,2}_{loc}(\mathbb{R}^n_+ \setminus \{0\})$. V satisfies

$$\sum_{k=1}^n \partial_k[-\partial_iV_k-\partial_kV_i+\frac{2}{n}divV\delta_{ik}]=0$$

weakly in $\mathbb{R}^n_+\setminus\{0\}$ for $i=1,\cdots,n$, and $V_n=\partial_nV_a=0$ on $\partial\mathbb{R}^n_+\setminus\{0\}$. By elliptic regularity theory, V is smooth in $\mathbb{R}^n_+\setminus\{0\}$. Thus, V satisfies $\Delta V_j + \frac{n-2}{n}\partial_j divV = 0$. This implies $\Delta divV = 0$. Moreover, on $\partial\mathbb{R}^n_+\setminus\{0\}$ we have $0=\Delta V_n + \frac{n-2}{n}\partial_n divV = 2\frac{n-1}{n}\partial_n\partial_nV_n$. Therefore, $\partial_n divV = 0$ on $\partial\mathbb{R}^n_+\setminus\{0\}$. We now define the function $divV$ on $\mathbb{R}^n\setminus\{0\}$ by standard reflection. Then $divV$ is a $C^{2,1}$ harmonic function in $\mathbb{R}^n\setminus\{0\}$. Since $\sup_{\mathbb{R}^n\setminus\{0\}}|x|^{2l}|divV|^2$ is bounded, we obtain $divV=0$ in $\mathbb{R}^n\setminus\{0\}$. Thus, $\Delta V_j=0$ in $\mathbb{R}^n_+\setminus\{0\}$. By the same reflection argument applied to the function V_a, we get V_a is a $C^{2,1}$ harmonic function in $\mathbb{R}^n\setminus\{0\}$. Since $\sup_{\mathbb{R}^n\setminus\{0\}}|x|^{2l-2}|V|^2<\infty$, we have $V_a=0$ in $\mathbb{R}^n\setminus\{0\}$. Finally, since $\partial_nV_n = divV = 0$, using the same reflection argument again we obtain $V_n=0$ in $\mathbb{R}^n\setminus\{0\}$. This contradicts to $\int_{(B_1\setminus B_{\frac{1}{2}})\cap\mathbb{R}^n_+}|V|^2dx>0$. Thus, (14) holds.

Now since we have

$$\begin{aligned}\int_{\mathbb{R}^n_+}((1+x_n)^2+\sum_{a=1}^{n-1}x_a^2)^{-n-2}|V|^2dx &\le C\int_{\mathbb{R}^n_+}((1+x_n)^2+\sum_{a=1}^{n-1}x_a^2)^{-n}|h|^2dx \\ &\le C\sum_{i,k=1}^n\sum_{|\alpha|=l}|h_{ik,\alpha}|^2,\end{aligned}$$

then by (14) $\sup_{r\ge1} r^{-2l-n-2}\int_{(B_{2r}\setminus B_r)\cap\mathbb{R}^n_+}|V|^2dx \le C\sum_{i,k=1}^n\sum_{|\alpha|=l}|h_{ik,\alpha}|^2$. Finally, by elliptic regularity $|\partial^\beta V|^2(x)\le C(n,|\beta|)\sum_{i,k=1}^n\sum_{|\alpha|=l}|h_{ik,\alpha}|^2(1+|x|)^{2|\alpha|+2-2|\beta|}$. $\square$

References

[1] Thierry Aubin. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl. (9)*, 55(3):269–296, 1976.

[2] William Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math. (2)*, 138(1):213–242, 1993.

[3] Simon Brendle. Convergence of the Yamabe flow for arbitrary initial energy. *J. Differential Geom.*, 69(2):217–278, 2005.

[4] Simon Brendle. Convergence of the Yamabe flow in dimension 6 and higher. *Invent. Math.*, 170(3):541–576, 2007.

[5] Simon Brendle. Blow-up phenomena for the Yamabe equation. *J. Amer. Math. Soc.*, 21(4):951–979, 2008.

[6] Simon Brendle and Szu-yu Sophie Chen. An existence theorem for the Yamabe problem on manifolds with boundary. preprint.

[7] Simon Brendle and Fernando C. Marques. Blow-up phenomena for the Yamabe equation II. *J. Differential Geom.*, 81(2):225–250, 2009.

[8] Szu-yu Sophie Chen. Conformal deformation on manifolds with boundary. *Geom. Funct. Anal.*, 19(4):1029–1064, 2009.

[9] Pascal Cherrier. Problèmes de Neumann non linéaires sur les variétés riemanniennes. *J. Funct. Anal.*, 57(2):154–206, 1984.

[10] José F. Escobar. Sharp constant in a Sobolev trace inequality. *Indiana Univ. Math. J.*, 37(3):687–698, 1988.

[11] José F. Escobar. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Ann. of Math. (2)*, 136(1):1–50, 1992.

[12] José F. Escobar. The Yamabe problem on manifolds with boundary. *J. Differential Geom.*, 35(1):21–84, 1992.

[13] José F. Escobar. Conformal metrics with prescribed mean curvature on the boundary. *Calc. Var. Partial Differential Equations*, 4(6):559–592, 1996.

[14] Lars Hörmander. *The analysis of linear partial differential operators. III.* Springer-Verlag, Berlin, 1985.

[15] M. A. Khuri, F. C. Marques, and R. M. Schoen. A compactness theorem for the yamabe problem. *J. Differential Geom.*, 81(1):143–196, 2009.

[16] Fernando C. Marques. Existence results for the Yamabe problem on manifolds with boundary. *Indiana Univ. Math. J.*, 54(6):1599–1620, 2005.

[17] Fernando C. Marques. Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary. *Comm. Anal. Geom.*, 15(2):381–405, 2007.

[18] Richard Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.*, 20(2):479–495, 1984.

[19] Neil S. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa (3)*, 22:265–274, 1968.

[20] Hidehiko Yamabe. On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.*, 12:21–37, 1960.

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